MATHEMATICAL MODEL OF ACOUSTIC FLUTTER OF SUPERSONIC GRILLS

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Transverse (flexural) vibrations of straight fins are observed in the supersonic grills of gas-dynamic CO_2 lasers (CO_2-GDL). These oscillations cause the gas flow at the entrance to the lasing region to be significantly nonuniform. There have even been cases of fatigue failure of the fins. In order to prevent such vibrations remote-controlled spacers are installed in the grill structure in the subsonic section of the interfin channel, but this gives rise to entropy nonuniformities owing to the dissipative processes accompanying flow past these spacers.

A qualitative explanation of one mechanism for excitation of vibrations of straight fins in the supersonic grill of a GDL is given in [1, 2]. It was assumed that the fins vibrate "rigidly" with the characteristic frequency. Then, under certain conditions the transverse force arising as a result of the excitation of longitudinal acoustic vibrations of the gas in the subsonic part of the channel has the same phase shift (relative to the phase of the fin vibrations), ensuring that energy flows from the gas flow to the fin. The transverse self-excited vibrations of the fins, due to the longitudinal acoustic vibrations of the gas, were termed acoustic flutter [1, 2].

In the present paper we formulate a more accurate mathematical model of acoustic flutter. We derive more accurate boundary conditions for the acoustic problem and, in contrast to [1, 2], we take into account the effect of the acoustic longitudinal waves on the fin motion itself (in the transverse direction). In other words, feedback is introduced into the system consisting of a fin (mechanical component) and acoustic vibrations of the gas (acoustic component). This model, in contrast to the preceding one, made it possible to determine the rate of growth of self-excited vibrations in it.

1. Formulation of the Acoustic Problem. As in [1, 2], we assume that the motion of a fin in the grill reduces to displacement of the fin as a whole in the direction of the y axis (see Fig. 1), and each fin vibrates in antiphase with the adjacent fin. In [1], it was established by means of numerical modeling that the contribution of the supersonic part of the interfin channel to exchange of energy between the gas flow and a fin is relatively small (less than 1%). This makes it possible to limit the theoretical analysis only to the subsonic part of the interfin channel. For this reason, the problem of acoustic flutter of the supersonic grill of a GDL can be formulated in the model setup as follows.

The subsonic part of the interfin channel (see Fig. 1) consists of a straight section of length L and height h_0 = const and a short tapered section ($\ell \ll L$). The critical (minimum) section of height h_{*0} is followed by the supersonic (expanding) part of the interfin channel. Let the transverse motion of the fin be given: $h(t) = h_0 + \Delta h(t)$. Our aim is to reduce the problem of nonstationary gas dynamics for the interfin channel to a linear problem for small disturbances on the rectilinear section ($0 < x < L$). The gas flow in this section with $h \ll L$ can be described by the system of equations of one-dimensional gas dynamics with time-varying cross section $h(t)$. The boundary condition on the pressure disurbance $p'(x, t)$ at $x = 0$ is $p'(0, t) = 0$.

Assume that for the section $[L, L + \ell]$ of the channel the condition of the flow is quasistationary. This happens when the characteristic frequencies of the perturbations of the gas parameters in the flow are much less than a/ℓ, where a is the velocity of sound in the gas. Then it turns out that a boundary condition can also be imposed at $x = L$ directly in front of the critical section of the interfin channel. We label the density ρ, the pressure p, the temperature T, and the gas flow velocity u with indices * and L, if they refer to the critical section $x = L + \ell$ and to the section $x = L$. We label with the index 0 quantities in the undisturbed flow and with a prime the perturbations of these quantities.

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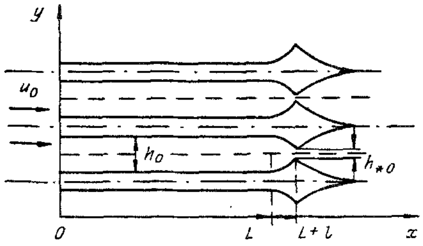

Fig. 1

Since the flow in the section L < x < L + ℓ is quasistationary, the condition that the flow rates be matched $(\rho uh)_* = (\rho uh)_L$ should be satisfied and there exist well-known expressions relating the values of the quantities in the critical section and in the section x = L:

$$u_*^2 = \frac{2}{\gamma-1} R\left(T + \frac{u^2}{2c_p}\right)_L,$$

$$\rho_* = \rho_L\left[\frac{\gamma+1}{2}\left(1 + \frac{(\gamma-1)u^2}{\gamma RT}\right)\right]_L^{1/(\gamma-1)}.$$

Hence, after corresponding transformations, we obtain the relations for the perturbations:

$$\left(\frac{\rho'}{\rho_0} + \frac{u'}{u_0}\right)_L = \left(1 - \frac{h_{*0}}{h_0}\right)\frac{\Delta h}{h_{*0}} + \frac{u_*'}{u_{*0}} + \frac{\rho_*'}{\rho_{*0}}; \tag{1.1}$$

$$\frac{u_*'}{u_{*0}} = \frac{1}{1+\frac{\gamma-1}{2}M_0^2}\left[\frac{T'}{T_0} + (\gamma-1)M_0^2\frac{u'}{u_0}\right]_L; \tag{1.2}$$

$$\frac{\rho_*'}{\rho_{*0}} = \left[\frac{\rho'}{\rho_0} + \frac{M_0^2}{1+\frac{(\gamma-1)}{2}M_0^2}\left(\frac{u'}{u_0} + \frac{T'}{2T_0}\right)\right]_L. \tag{1.3}$$

Here γ is the adiabatic index of the gas; R is the gas constant; and $M_0 = u_0/a$ is the Mach number (usually, $M_0 \ll 1$).

Assuming that the processes are adiabatic and taking into account the equation of state of the gas p = ρRT, we find from Eqs. (1.1)-(1.3) the boundary condition for the section x = L:

$$\alpha u'/u_0 - \beta p'/p_0 = \eta\Delta h/h_{*0},$$

where $\alpha = 1 - 2\gamma M_0^2/(2+(\gamma-1)M_0^2)$; $\beta = (1-1/\gamma)(2-M_0^2)/(2+(\gamma-1)M_0^2)$; $\eta = (1-h_{*0}/h_0)$.

2. Analysis of the Mathematical Model. The system of equations of motion of the gas (equations of acoustics in a moving gas), linearized with respect to the perturbations, in the formulation of the problem under consideration and the boundary conditions are as follows (the primes are dropped here and below):

$$\partial u/\partial t + u_0\partial u/\partial x + (1/\rho_0)\partial p/\partial x = 0; \tag{2.1}$$

$$\partial p/\partial t + u_0\partial p/\partial x + \gamma p_0\partial u/\partial x = 0; \tag{2.2}$$

$$p(0,\ t) = 0; \tag{2.3}$$

$$\alpha u\,(L, t)/u_0 - \beta p\,(L, t)/p_0 = \eta \Delta h\,(t)/h_{*0}. \tag{2.4}$$

In the acoustics equations (2.1) and (2.2) the terms containing $\dot{h}(t)/h_0$, related with the change in the cross section of the channel, are omitted; the contribution of these terms to the excitation of acoustic vibrations is small compared with the contribution of the right-hand term in the boundary condition (2.4).

Let the equation for the change in the gap $z(t) = \Delta h(t)$ describe the mechanical component of our model:

$$m\ddot{z} + kz = f, \tag{2.5}$$

where

$$f = 2H \int_0^L p\, dx \tag{2.6}$$

is the lateral force exerted by the perturbation of the gas pressure on the fin; m, k, and H are the mass, spring constant, and width, respectively, of a fin.

We seek the solution of the system of equations (2.1)-(2.6) in the form $u(t, x) = U(x) \exp(\lambda t)$, $p(t, x) = P(x) \exp(\lambda t)$, $z(t) = Z \exp(\lambda t)$, $f(t) = F \exp(\lambda t)$ (λ is a complex number). Then we write the system (2.1)-(2.6) as

$$\lambda U + u_0 \partial U/\partial x + (1/\rho_0)\partial P/\partial x = 0; \tag{2.1'}$$

$$\lambda P + u_0 \partial P/\partial x + \gamma p_0 \partial U/\partial x = 0; \tag{2.2'}$$

$$P(0) = 0; \tag{2.3'}$$

$$\alpha U\,(L)/u_0 - \beta P\,(L)/p_0 = \eta Z/z_0 \quad (z_0 = h_{*0}); \tag{2.4'}$$

$$\lambda^2 mZ + kZ = F; \tag{2.5'}$$

$$F = 2H \int_0^L P\, dx. \tag{2.6'}$$

The solution of the system (2.1') and (2.2') has the form

$$U = A_1 \exp(\mu_1 x) + A_2 \exp(\mu_2 x),\ P = C_1 \exp(\mu_1 x) + C_2 \exp(\mu_2 x). \tag{2.7}$$

Here

$$\mu_1 = \lambda/(a - u_0),\ \mu_2 = -\lambda/(a + u_0) \tag{2.8}$$

are found from Eqs. (2.1') and (2.2'). Taking into account the boundary conditions (2.3') and (2.4') we obtain

$$C_1 = -C_2 = -\eta Z z_0^{-1} \rho_0 a^2 M/g\,(\lambda), \tag{2.9}$$

where

$$g(\lambda) = \exp(\mu_1 L) + \varkappa \exp(\mu_2 L); \tag{2.10}$$

$$\varkappa = (\alpha - \gamma\beta M)/(\alpha + \gamma\beta M), \; M = u_0/a. \tag{2.11}$$

According to Eqs. (2.6') and (2.7),

$$F = 2HC_1(\exp(\mu_1 L) - 1)/\mu_1 + 2HC_2(\exp(\mu_2 L) - 1)/\mu_2. \tag{2.12}$$

Substituting Eqs. (2.9) and (2.12) into Eq. (2.5'), we find an equation for the eigenvalues λ:

$$G(\lambda) = \lambda^2 + \Omega^2 + \varepsilon\varphi(\lambda)/g(\lambda) = 0. \tag{2.13}$$

Here

$$\begin{gathered}\varphi(\lambda) = (\exp(\mu_1 L) - 1)/\mu_1 - (\exp(\mu_2 L) - 1)/\mu_2, \\ \Omega = \sqrt{k/m}, \quad \varepsilon = 2H\eta a^2 M\rho_0 z_0^{-1} m^{-1}(\alpha + \gamma\beta M)^{-1}.\end{gathered} \tag{2.14}$$

Self-excitation in the model under consideration is observed only if the function $G(\lambda)$ is equal to zero in the right-hand half-space of the complex variable λ. The root with the largest real component determines the rate of growth of the self-excitation.

We first find the poles of the function $G(\lambda)$. From Eqs. (2.8) and (2.10) we obtain the following expression for all roots of the function $g(\lambda)$:

$$\lambda_n = \frac{a(1 - M^2)}{2L}[\ln\varkappa + i\pi(2n + 1)], \; n = \ldots, -1, 0, 1, \ldots \tag{2.15}$$

All these roots are first order. We note that

$$\varphi(\lambda)\mu_1 - g(\lambda) = [(1 + M)/((1 - M) - \varkappa)]\exp(\mu_2 L) - 2/(1 - M). \tag{2.16}$$

All points at which the right-hand side of Eq. (2.16) vanishes are expressed by the formula

$$\lambda'_m = (a/L)(1 + M)(\ln\varkappa_1 + 2\pi i m), m = \ldots, -1, 0, 1, \ldots, \tag{2.17}$$

where

$$\varkappa_1 = 0{,}5[1 + M - \varkappa(1 - M)]. \tag{2.18}$$

We see from Eqs. (2.11), (2.15), (2.17), and (2.18) that ($M \ll 1$) the inequality $\operatorname{Re}\lambda_n > \operatorname{Re}\lambda'_m$, is satisfied, i.e., the sets (2.15) and (2.17) do not have common points. For this reason, it follows from Eq. (2.16) that the roots of the function $g(\lambda)$ are not the roots of the function $\varphi(\lambda)$. The function $\varphi(\lambda)$ is regular in the entire complex plane of the variable λ. Therefore, the points λ_n determined by the formula (2.15) and only these points are poles of the function $G(\lambda)$.

For arbitrary λ, $\delta > 0$ we designate by $O(\lambda, \delta)$ a circle of radius δ centered at the point λ. Let $\delta_1 = 0.5|\lambda_2 - \lambda_1|$ and $\delta_0 = 0.5 \min\limits_n |\lambda_n - i\Omega| > 0$. It follows from the function $G(\lambda)$ that for $\delta = \min(\delta_0, \delta_1)$ and sufficiently small ε outside the region

$$Q_\delta = \bigcup_{n=-\infty}^{n=+\infty} O(\lambda_n, \delta) \cup O(i\Omega, \delta) \cup O(-i\Omega, \delta)$$

this function will not have any roots.

According to Eq. (2.13), the roots of the function $G(\lambda)$ satisfy the equations $\lambda = R_{\pm}(\lambda)$, $\lambda = R_n(\lambda)$, $n = ..., -1, 0, 1, ...,$ where

$$R_n(\lambda) = \lambda_n - \varepsilon \frac{\varphi(\lambda)}{(\lambda^2 + \Omega^2)\, y_n(\lambda)}; \tag{2.19}$$

$$R_{\pm}(\lambda) = \pm i\Omega - \varepsilon \frac{\varphi(\lambda)}{(\lambda \pm i\Omega)\, g(\lambda)}. \tag{2.20}$$

Here $y_n(\lambda) = g(\lambda)/(\lambda - \lambda_n)$ is a regular function in the region $O(\lambda_n, \delta)$, since $g(\lambda)$ has a first order root at the point λ_n. For sufficiently small ε the functions $R_{\pm}(\lambda)$ and $R_n(\lambda)$ realize a contraction of the regions $O(\pm i\Omega, \delta)$, $O(\lambda_n, \delta)$ $(n = ..., -1, 0, 1, ...)$ into themselves, respectively. Thus, to a first approximation in ε, the roots of the function $G(\lambda)$ will be given by the formulas

$$N_n \approx \lambda_n - \varepsilon \frac{\varphi(\lambda_n)}{(\lambda_n^2 + \Omega^2)\, y_n(\lambda_n)}, \qquad N_{\pm} \approx \pm i\Omega - \varepsilon \frac{\varphi(\pm i\Omega)}{\pm 2i\Omega g(\pm i\Omega)}. \tag{2.21}$$

Since $\varkappa < 1$, according to Eq. (2.15), Re λ_n = const < 0 for all n. For this reason, for sufficiently small ε only the roots $N_{\pm}$ can have a positive real component.

Substituting into the formula (2.21) the expressions (2.10) and (2.14) and then Eq. (2.8), we find $\operatorname{Re} N_{\pm} \approx 0.5\varepsilon a \Omega^{-2} D/E$, where

$$D = [(1 - M) + \varkappa(1 + M)] + [(1 + M) + \varkappa(1 - M)] \cos \frac{2Y}{1 - M^2} - 2\cos\frac{Y}{1 - M} - 2\varkappa \cos\frac{Y}{1 + M}; \quad E = (1 - \varkappa)^2 + 4\varkappa\left(\cos\frac{2Y}{1 - M^2}\right)^2; \quad Y = \Omega L/a.$$

For M << 1

$$\frac{D}{E} \approx \frac{2(1 + \varkappa)\cos Y(\cos Y - 1)}{(1 - \varkappa)^2 + 4\varkappa \cos^2 Y}, \tag{2.22}$$

whence for the maximum possible rate of growth of self-excitation (for fixed $\varkappa$) we obtain the expression

$$\operatorname{Re} N_{\pm} \approx 0{,}5\varepsilon a\Omega^{-2}/(1 - \varkappa) = 0{,}5H\eta\alpha^2\beta^{-1}\gamma^{-1} a^3\rho_0 z_0^{-1} m^{-1}\Omega^{-2}.$$

According to Eq. (2.22), Re $N_{\pm} > 0$ for $0.5\pi + 2\pi m < Y < 1.5\pi + 2\pi m$, $m = 0, 1, 2...$. Determining a more accurate value of D to second order in M inclusively ($(1 - \varkappa) \sim M$), we have $D \approx 2(1 + \varkappa)\cos Y(\cos Y - 1) + 2M(1 - \varkappa)(Y \sin Y - \sin^2 Y)$. Hence we find more accurately the intervals of values of Y in which Re $N_{\pm} > 0$: $0.5\pi - \tau(0.5\pi - 1 + 2\pi m) + 2\pi m < Y < 1.5\pi + \tau(1.5\pi - 1 + 2\pi m) + 2\pi m$. Here $\tau = M(1 - \varkappa)/(1 + \varkappa)$. The expression for these intervals is valid for $2\pi m\tau \ll 1$, $m = 0, 1, \ldots$.

Remark. For small values of δ_0 the requirement that ε be small becomes more stringent. However it can be weakened, if instead of the functions (2.19) and (2.20) we study the mapping

$$\frac{\lambda_k \pm i\Omega}{2} + \sqrt{\frac{(\lambda_k \mp i\Omega)^2}{4} - \varepsilon \frac{\varphi(\lambda)}{(\lambda \pm i\Omega)\, y_k(\lambda)}},$$

where k designates the values for which $|\lambda_k \pm i\Omega| = 2\delta_0$. The stationary points of such a mapping are also roots of the function $G(\lambda)$. With the help of the expression (2.23) it can be proved that for some c, $\varepsilon_0 > 0$ with $\varepsilon < \varepsilon_0$ the rate of growth of the self-excitation is estimated above, uniformly in $\delta_0 > 0$, by the quantity $c\sqrt{\varepsilon}$.

LITERATURE CITED

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